On Monetary Policy, Model Uncertainty, and Credibility

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Abstract

This paper studies the design of optimal time-consistent monetary policy in an economy where the planner and a representative household are faced with model uncertainty: While they are able to construct and agree on a reference model (probability distribution) governing the evolution of the exogenous state of the economy, a representative household has fragile beliefs and is averse to model uncertainty. In such environments, management of households' inflation expectations becomes an active channel of optimal policymaking per se. A central banker who respects the fact that private sector models are imperfect and designs her optimal policy accordingly may be able not only to mitigate a fundamental time-inconsistency problem but also to sustain higher welfare. Interestingly, in some cases the resulting welfare is even higher than in models where both a central

banker and a representative household are assumed to know the true model, i.e., to have rational expectations.

1 Introduction

The idea that monetary policy is primarily about "managing expectations" is well accepted amongst the central bankers around the world and monetary policy theorists alike. Indeed, many leading monetary economists see the management of expectations as *the* task for monetary policy. For Svensson (2004), *"Monetary policy is to a large extent the management of expectations,"* while according to Woodford (2005), *"Not only do expectations about policy matter, but, at least under current conditions, very little else matters."* But what do we mean precisely by the "management of expectations?" And what is the right framework to analyze this concept?[1] Sargent (2022) suggests that *"To make progress on this topic requires a setting in which, first, private agents and the government have different beliefs, and second, the government has a model of how its actions affect private agents beliefs, and third, discrepancies of beliefs between government and private agents can be rationalized. Filling all three of these requirements simultaneously is a tall order."* Doing so is the contribution of this paper.

Under rational expectations (a common beliefs framework), a government strategy plays two roles, as noted by Sargent (2008).[2] First, it is the actual policy rule, say setting the policy interest rate. Second, it is a system of private expectations about that very policy. The theory is silent about who chooses that equilibrium system of private-sector beliefs. Instead, the theory is about how – confronted with a given system of expectations – the policymaker "chooses" to confirm them.

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[1]These questions are subject of Bernanke (2007). In particular, the author notes that *"The traditional rational-expectations model of inflation and inflation expectations has been a useful workhorse for thinking about issues of credibility and institutional design, but, to my mind, it is less helpful for thinking about economies in which (1) the structure of the economy is constantly evolving in ways that are imperfectly understood by both the public and policymakers and (2) the policymakers' objective function is not fully known by private agents. In particular, together with the assumption that the central bank's objective function is fixed and known to the public, the traditional rational-expectations approach implies that the public has firm knowledge of the long-run equilibrium inflation rate; consequently, their long-run inflation expectations do not vary over time in response to new information."* After eight years of serving as the Chairman of the Federal Reserve System, Bernanke (2015) wrote on his blog *"When I was at the Federal Reserve, I occasionally observed that monetary policy is 98 percent talk and only 2 percent action."*

[2]Throughout this paper I use the terms "policymaker," "government," "planner," and "central banker" interchangeably to refer to the agent responsible for the setting of the optimal monetary policy in the model economy.

One way to abstract from the rational expectations paradigm and the common beliefs assumption is through the assumption of incomplete information, where the policymaker and the public have access to potentially different information (Cukierman and Meltzer (1986), Cogley, Matthes, and Sbordone (2015), Lorenzoni (2010)). In this strand of literature, it is typically still true that both the private agents and the planner have unique but not common priors or systems of beliefs. Then, depending on the information (e.g., public signals) that the government chooses to communicate to the agents she can affect how quickly the agents are learning, and, in that sense, at least in the short run, affect their beliefs.

Yet another way to think about why private agents' and government beliefs might differ is as a result of model uncertainty: While both the public and the planner might have access to the same information, they may entertain alternative probability distributions as candidate data generating processes. As this paper demonstrates, in such a framework the central bank's management of private beliefs becomes an integral part of the theory of optimal conduct of monetary policy.

In the model economy, constructed in the tradition of monetary models by Calvo (1978) and Chang (1998), a representative household derives utility from consumption and real money holdings. The government uses the newly printed money to finance transfers or taxes to households. Taxes and transfers are distortionary. The only source of uncertainty in this economy is a shock that affects the degree of tax distortions through its influence on households' income.

At the heart of this paper lies the assumption that the government has a *single reference model* that describes the evolution of the underlying fundamental shock while a representative household has beliefs which are fragile: She fears that this reference model might be misspecified. As in Hansen and Sargent (2008), to confront this concern, the representative household contemplates a set of nearby probability distributions (models) and seeks decision rules that would work well across these models. The household assesses the performance of a given decision rule by computing the expected utility under the worst-case distribution within the set. This worst-case distribution can be seen as the outcome that follows from twisting the probabilities under the reference model with adequate (endogenous) probability distortions.

As for the second part of Sargent (2022)'s "tall order" above, in the model, the government recognizes that households are not able or willing to assign a unique prob-

ability distribution to alternative realizations of the stochastic state of the economy. The government wants to design optimal monetary policy that explicitly accounts for the fact that households' allocation rules are influenced by how they form their beliefs in light of model uncertainty.

I characterize optimal policy under two timing protocols for the government's choices. First, I work under the assumption that at time zero the government can commit to a policy specifying its actions for all current and future dates and states of nature. Under this assumption, at time zero a government chooses the best competitive equilibrium from the set of competitive equilibria with model uncertainty, i.e. one that maximizes the households' expected lifetime utility but under the government's own unique beliefs. I will refer to such a government as *paternalistic Ramsey planner*.

The competitive equilibrium conditions in this model are represented by the households' Euler equations and an exponential twisting formula for the probability distortions (in a representative agent's beliefs with respect to planner's beliefs). Using insights from Kydland and Prescott (1980), I express the competitive equilibria in a recursive structure by introducing an adequate pair of state variables. I first need to keep track of the equilibrium (adjusted) marginal utilities to guarantee that the Euler equations are satisfied after each history. The second state variable is the households' lifetime utility. This variable is needed to express the equilibrium probability distortions in the context of model uncertainty. These two variables summarize all the relevant information about future policies and allocations for households' decisionmaking when the government has the ability to commit. Through the dynamics of the promised marginal utility and households' value (lifetime utility), which the government has to deliver in equilibrium, the solution to the government's problem under commitment, the Ramsey plan, exhibits history dependence.

Once I relax the assumption that the government has the power to commit but instead chooses sequentially, a time inconsistency problem may arise, as first noted by Kydland and Prescott (1977) and Calvo (1978). The government will adhere to a plan only if it is in its own interest to do so. As a consequence, it is urgent to check whether the optimal policies derived by the paternalistic Ramsey planner are time consistent, and, more generally, to characterize the set of *sustainable plans with model uncertainty*.[3] This latter notion should be thought of as an extension of Chari and

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[3]The notion of a sustainable plan inherits sequential rationality on the government's side, jointly

Kehoe (1990). Using the government's value as an additional third state variable, an appropriate incentive constraint for the government can then be constructed to complete the formulation of a sustainable plan in a recursive way. This introduces a new source of history-dependence given by the restrictions that the system of households' expectations impose on the government's policy actions in equilibrium.

To my knowledge, this paper constitutes the first attempt to characterize the set of all time-consistent outcomes when private agents are faced with model uncertainty and their beliefs are endogenously distorted with respect to the planner's beliefs. The fact that the extent to which a representative household's beliefs are misspecified with respect to planner's beliefs is determined endogenously in equilibrium speaks to the third element of the Sargent (2022)'s "tall order": Because private agents choose policies that are best responses to their worst-case model (rather than the shared reference model) – which is an object that is affected by the planners policy (whether Ramsey or as part of a sustainable plan, defined below) – the planner is thrust into manipulating private agents beliefs and the theory of rationalizing discrepancies in beliefs between private agents and a planner becomes an integral part of a theory of the management of expectations.

Characterizing time-consistent outcomes is a challenging task because any time-consistent solution must include a description of government and market behavior such that the continuation of such behavior after any history is a competitive equilibrium and it is optimal for the government to follow that policy. In this paper, I use insights from the work by Abreu, Pearce, and Stacchetti (1990), Chang (1998), and Phelan and Stacchetti (2001) to compute the sets of equilibrium payoffs as the largest fixed point of an appropriate operator. Previously, numerical examples in Orlik and Presno (2013) suggested that government policies that account for the fact that households contemplate a set of probability distributions may lead to better outcomes. Here, (1) I clarify the conditions under which these numerical solutions are valid; (2) I show analytically why Ramsey planner in an economy with uncertainty-averse representative household obtains higher welfare (Theorem 1); (3) I solve a simplified three-period version of the model to explain why aversion towards model uncertainty on the side of households can help the government mitigate the time-consistency problem; (4) and

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with the fact that households always respond to government actions by choosing from competitive equilibrium allocations.

I present an extension of the baseline model with two-sided model uncertainty and conjecture mechanisms at play and conditions under which the same qualitative results obtain.

Although in this paper I restrict attention to the type of models of monetary policy-making that can be cast in the spirit of Calvo (1978), the approach could be applicable to many repeated or dynamic games between a government and a representative household who distrusts the model used by the government.

To my knowledge, there is only a handful of papers that try to explore the policymaker's role in managing households' expectations in the presence of model uncertainty. Karantounias (2013) studies the optimal fiscal policy problem in Lucas and Stokey (1983) but in an environment where a representative household distrusts the model governing the evolution of exogenous government expenditures. Karantounias (2013) applies the techniques of Marcet and Marimon (2009) to characterize the optimal policies when the government has power to commit. Woodford (2010) discusses the optimal monetary policy under commitment in an economy where both the government and the private sector fully trust their own models, but the government distrusts its knowledge of the private sector's beliefs about prices. Adam and Woodford (2021) analyze optimal monetary policy in a New Keynesian model with housing: When the policymaker is concerned with potential departures of private sector expectations from rational ones and seeks a policy that is robust against such possible departures, then the optimal target criterion must also depend on housing prices.

This paper is also related to the literature which demonstrates that beliefs of various agents may be inconsistent with the assumption of unique probability distributions. The fact that private agents seem unable to assign a unique probability distribution to alternative outcomes has been demonstrated in a seminal work of Ellsberg (1961) and similar experimental studies, e.g. Halevy (2007). A lack of confidence in the singular models seems to have become apparent during the recent financial crisis (Caballero and Krishnamurthy (2008), Uhlig (2010), Boyarchenko (2012)). Bhandari, Borovička, and Ho (2022) argue that households' inflation and unemployment forecasts in the Michigan Survey can be well explained assuming households are endowed with a version of multiplier preferences I consider here.

This paper is also related to game-theoretical studies which analyze comparative statics with respect to players' risk aversion or ambiguity aversion. In Battigalli et al.

(2016) higher ambiguity aversion is shown to expand the set of justifiable actions (i.e., best-reply actions to some belief over probabilistic models). Under ambiguity neutrality, higher risk aversion expands the set of justifiable actions. Similarly, for subjective expected utility maximizers in finite games, Weinstein (2016) shows that the rationalizable set grows with increased risk aversion. While these papers study the consequences of increased risk aversion and / or ambiguity aversion in the space of players' actions, in this paper I focus on the resulting lifetime utility values which can be sustained when agents are ambiguity averse.

The remainder of this paper is organized as follows. Section 2 sets up the model and outlines the assumptions made. Section 3 introduces the notion of competitive equilibrium with model uncertainty. Section 4 discusses the recursive formulation of the Ramsey problem for the paternalistic government. Section 5 contains the discussion of sustainable plans with model uncertainty. Section 6 solves a simplified three-period version of the model that sheds light on the main mechanism by which aversion to model uncertainty on the side of households may substitute for the lack of government commitment. Section 7 briefly discusses an extension of the model with both the government and households averse to model uncertainty. Finally, Section 8 concludes.

2 Benchmark Model

The model economy is populated by two infinitely lived agents: a representative household (with her alter ego, which represents her concerns about model misspecification) and a government. The household and the government interact with each other at discrete dates indexed as $t = 0, 1, ...$

At the beginning of each period, the economy is hit by an exogenous shock. The government in the model has a reference or approximating probability model for this shock, which is its best estimate of the economic dynamics.[4] While the government fully trusts the probability distribution for the shock, the representative household fears that it is misspecified. In turn, she contemplates a set of alternative probability distributions to be endogenously determined, and seeks decision rules that perform well over this set of distributions. Given her fragile beliefs - doubts regarding which model

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[4]Throughout the paper, I use the terms "probability model" and "probability distribution" interchangeably.

actually governs the evolution of the shock - the household designs decision rules that guarantee lower bounds on expected utility level under any of the distributions.

Let $(\Omega, \mathcal{F}, \mathrm{Pr})$ be the underlying probability space. Let the exogenous shock be given by s_t, where $s_0 \in \mathbb{S}$ is given (there is no uncertainty at time 0) and $s_t : \Omega \to \mathbb{S}$ for all $t > 0$. The set $\mathbb{S}$ for the shock is assumed to be finite with cardinality S. I assume that s_t follows a Markov process for all $t > 0$, with transition probabilities given by $\pi(s_{t+1}|s_t)$.

Throughout this paper I will refer to the conditional distribution $\pi(s_{t+1}|s_t)$ as the *reference model*. Let $s^t \equiv (s_0, s_1, ..., s_t) \in \mathbb{S} \times \mathbb{S} \times ... \times \mathbb{S} \equiv \mathbb{S}^{t+1}$ be the history of the realizations of the shock up to t. Finally, I denote by $\mathcal{S}^t \equiv \mathcal{F}(s^t)$ the sigma-algebra generated by the history s^t.

2.1 The Representative Household's Problem and Fears about Model Misspecification

The households in this economy derive utility from consumption of a single good, $c(s^t)$, and real money balances, $m(s^t)$. The household's period payoff is given by $u(c_t(s^t)) + v(m_t(s^t))$, where the utility components u and v satisfy the following assumptions:

[A1] $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, and strictly concave

[A2] $v : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, and strictly concave

[A3] $\lim_{c \to 0} u'(c) = \lim_{m \to 0} v'(m) = +\infty$

[A4] $\exists \overline{m} < +\infty$ such that $v'(\overline{m}) = 0$.

The assumptions [A1]-[A3] are standard. Assumption [A4] defines a satiation level for real money balances.

In this paper, I model the representative household as being uncertainty-averse. While the government fully trusts the reference model $\pi(s^t)$, the household distrusts it. For this reason, she surrounds it with a set of alternative distributions $\widetilde{\pi}(s^t)$ that are statistical perturbations of the reference model, and seeks decision rules that perform well across these alternative distributions. I assume that these alternative distributions,

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$\widetilde{\pi}(s_t)$, are absolutely continuous with respect to $\pi(s_t)$, i.e. $\pi(s_t) = 0 \Rightarrow \widetilde{\pi}(s^t) = 0$, $\forall s^t \in \mathbb{S}^{t+1}$.

By invoking the Radon-Nikodym theorem I can express any of these alternative distorted distributions using a nonnegative $\mathcal{S}^t$-measurable function given by

$$D_t\left(s^t\right) = \begin{cases} \frac{\widetilde{\pi}\left(s^t\right)}{\pi\left(s^t\right)} & \text{if } \pi\left(s^t\right) > 0 \\ 1 & \text{if } \pi\left(s^t\right) = 0, \end{cases}$$

which is a martingale with respect to $\pi\left(s^t\right)$, i.e. $\sum_{s_{t+1}} \pi\left(s_{t+1}|s_t\right) D_{t+1}\left(s^{t+1}\right) = D_t\left(s^t\right)$.

I can also define the conditional likelihood ratio as $d_{t+1}\left(s_{t+1}|s^t\right) \equiv \frac{D_{t+1}\left(s^t, s_{t+1}\right)}{D_t\left(s^t\right)}$ for $D_t\left(s^t\right) > 0$. Notice that in case $D_t\left(s^t\right) > 0$ it follows that

$$d_{t+1}\left(s_{t+1}|s^t\right) = \begin{cases} \frac{\widetilde{\pi}\left(s_{t+1}|s^t\right)}{\pi\left(s_{t+1}|s_t\right)} & \text{if } \pi\left(s^{t+1}\right) > 0 \\ 1 & \text{if } \pi\left(s^{t+1}\right) = 0, \end{cases}$$

and that the expectation of the conditional likelihood ratio under the reference model is always equal to 1, i.e. $\sum_{s_{t+1}} \pi\left(s_{t+1}|s^t\right) d_{t+1}(s_{t+1}|s^t) = 1$.

To express the concerns about model misspecification, I follow Hansen and Sargent (2008) and endow the household with multiplier preferences. In this case, the set of alternative distributions over which the household evaluates the expected utility of a given decision rule is given by an entropy ball. I can then think of the household as playing a zero-sum game against her *alter ego*, who is a fictitious agent that represents her fears about model misspecification. The alter ego will be distorting the expectations of continuation outcomes in order to minimize the household's lifetime utility. She will do so by selecting a worst-case distorted model $\widetilde{\pi}\left(s^t\right)$, or equivalently, a sequence of probability distortions $\{D_t\left(s^t\right), d_{t+1}\left(s_{t+1}|s^t\right)\}_{t=0}^{\infty}$.

The representative household ranks contingent plans for consumption and money balances according to

$$V^H = \max_{\{c_t(s^t), m_t(s^t)\}} \min_{\{D_t(s^t), d_{t+1}(s_{t+1})\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) D_t(s^t) \left\{ \left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right))\right] \right.$$

$$\left. + \theta\beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) \log d_{t+1}(s_{t+1}|s^t) \right\} \tag{1}$$

$$D_{t+1}(s^{t+1}) = d_{t+1}(s_{t+1}|s^t) D_t(s^t) \tag{2}$$

$$\sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) = 1, \tag{3}$$

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where $m_t \equiv q_t M_t$ is the real money balances, M_t is the money holdings at the end of period t, q_t is the value of money in terms of the consumption good (that is, the reciprocal of the price level), and $\theta \in (\underline{\theta}, +\infty]$ is a penalty parameter that controls the degree of concern about model misspecification. Through the last term, the entropy term, the alter ego is being penalized whenever she selects a distorted model that differs from the approximating one. Note that the higher the value of θ, the more the alter ego is being punished. Allowing $\theta \to +\infty$, the probability distortions to the approximating model vanish, the household and the government share the same beliefs, and expression (1) collapses to the standard expected utility.

Conditions (2) and (3) discipline the choices of the evil alter ego. Condition (2) defines recursively the likelihood ratio D_t. Condition (3) guarantees that every distorted probability is a well-defined probability measure.

The minimization problem yields lower bounds (in terms of expected utility) on the performance of any of the household's decision rules. The probability distortion $d(s_{t+1}|s^t)$ that solves this minimization problem satisfies the following exponential twisting formula

$$d(s_{t+1}|s^t) = \frac{\exp\left(-\frac{V^H(s^{t+1})}{\theta}\right)}{\sum_{s_{t+1} \in \mathbb{S}} \pi(s_{t+1}|s_t) \exp\left(-\frac{V^H(s^{t+1})}{\theta}\right)}, \tag{4}$$

where $V^H(s^{t+1})$ is the $t+1-$equilibrium value for the household. Condition (4) shows how the alter ego pessimistically twists the household's beliefs by assigning high probability distortions to the states s_{t+1} associated with low utility for the household, and low probability distortions to the high-utility states. See the Appendix A.1 for the derivation of condition (4). Notice from (4) that to express the optimal belief distortions, one needs to know the household's equilibrium values. Using expression (4) the expected lifetime utility of the household at time t, in equilibrium, is

$$V(s^t) = u(c(s^t)) + v(m(s^t)) - \beta\theta \log \sum_{s_{t+1} \in \mathbb{S}} \pi(s_{t+1}|s_t) \left(\exp\left(-\frac{V^H(s^{t+1})}{\theta}\right)\right).$$

The representative household takes sequences of prices, $\{q_t(s^t)\}_{t=0}^{\infty}$, income, $\{y_t(s^t)\}_{t=0}^{\infty}$, taxes or subsidies, $\{x_t(s^t)\}_{t=0}^{\infty}$, and the conditional likelihood ratio, $\{d_{t+1}(s_{t+1}|s^t)\}_{t=0}^{\infty}$, as well as the initial money supply M_{-1}, shock realization s_0 and $D_0 = 1$, as given.

The household then maximizes (1) subject to the following constraints

$$q_t\left(s^t\right) M_t\left(s^t\right) \leq y_t\left(s^t\right) - x_t\left(s^t\right) - c_t\left(s^t\right) + q_t\left(s^t\right) M_{t-1}(s^{t-1}) \tag{5}$$

$$q_t\left(s^t\right) M_t\left(s^t\right) \leq \overline{m}. \tag{6}$$

Condition (5) represents the household's budget constraint, which states that for all $t \geq 0$ and all s^t after-tax income in period t, $y_t - x_t$, together with the value of money holdings carried from last period, must be sufficient to cover the period-t expenditures on consumption and new purchases of money. Condition (6) is introduced for technical reasons, in order to bound real money balances from above (so that the optimum quantity of money in this economy be well defined, as in Calvo (1978)).

2.2 Government

In this economy the government chooses how much money, $M_t(s^t)$ to create or to withdraw from circulation. In particular, it chooses a sequence $\{h_t\}_{t=0}^{\infty}$ where h_t is the reciprocal of the gross rate of money growth for all $t \geq 0$, i.e. $h_t \equiv \frac{M_{t-1}}{M_t}$. I make the following assumption on the set of values for the inverse money growth rate,

[A5] $h_t(s^t) \in \Pi \equiv [\underline{\pi}, \overline{\pi}]$ with $0 < \underline{\pi} < 1 < \frac{1}{\beta} \leq \overline{\pi}$.

[A5] establishes *ad hoc* bounds on the admissible rates for money creation. A positive lower bound implies that the supply of money has to be positive. The upper bound is set for technical reasons.

The government runs a balanced budget by printing money to finance the transfers to households or destroying the money it collects in the form of tax revenues, x_t,

$$x_t(s^t) = q_t(s^t)\left[M_{t-1}\left(s^{t-1}\right) - M_t(s^t)\right]. \tag{7}$$

Using the definition of m_t and h_t, (7) can be reformulated as

$$x_t(s^t) = m_t(s^t)\left[h_t(s^t) - 1\right]. \tag{8}$$

Notice that from equation (8) $x_t(s^t) \in \mathbb{X} \equiv \left[(\underline{\pi} - 1)\,\overline{m}, (\overline{\pi} - 1)\,\overline{m}\right]$.

As in Chang (1998), I assume that taxes and subsidies are distortionary. To model that, I consider an *ad hoc* functional form for households' income, $f : \mathbb{X} \times \mathbb{S} \to \mathbb{R}$, that depends on tax collections in period t and the exogenous shock, s_t, i.e. $y_t(s^t) \equiv$

$f(x_t(s^t), s_t)$.[5] The function $f : \mathbb{X} \times \mathbb{S} \to \mathbb{R}$ is assumed to be at least twice continuously differentiable with respect to its first argument and

[A6] $f(x, s) > 0$, $f_1(0, s) = 0$, $f_{11}(x, s) < 0$ for all $x \in \mathbb{X}$, for all $s \in \mathbb{S}$

[A7] $f(x, s) = f(-x, s) > 0$ for all $x \in \mathbb{X}$, for all $s \in \mathbb{S}$,

where f_1 and f_{11} denote, respectively, the first and second derivative of function f with respect to its first argument. Function f is intended to convey that taxes (and transfers) are distortionary without the need to model the nature of such distortions explicitly. [A6] indicates that income is always positive and that it is increasingly costly in terms of consumption to set taxes or to make transfers to households. This assumption will play a key role in the time-inconsistent nature of the *Ramsey plan*, when the government can commit to its announced policies. The symmetry of f given by [A7] implies that taxes and subsidies are equally distortionary, for simplicity.

2.3   The Within-Period Timing Protocol

The timing protocol within each period is as follows. First, the shock realization, $s_t(s^{t-1})$, occurs. Then, the government observes the shock, chooses the money supply growth rate $h_t(s^t)$ and taxes $x_t(s^t)$ for the period, and announces a sequence of future money growth rates and tax collections $\{h_{t+1}(s^{t+1}), x_{t+1}(s^{t+1})\}_{t=0}^{\infty}$. After that, given prices $q_t(s^{t-1})$, the current policy actions $(h_t(s^t), x_t(s^t))$ and their expectations of future policies, the household chooses $M_t(s^{t-1})$, or equivalently real balances $m_t(s^t)$. When making her choice of $m_t(s^t)$, the household can be thought of as playing a zero-sum game against her alter ego, who distorts her beliefs' about the evolution of future shock realizations.[6] Then taxes are collected and output is realized, $y_t(s^t) = f(s_t(s^{t-1}), x_t(s^t))$. Finally, consumption $c_t(s^t)$ takes place.

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[5]The formulation of the f function can be thought of as reflecting the ad hoc tax collection costs in Barro (1979). The shock s may shift the income level irrespective of the tax x. In addition, it may affect how much output falls when the government levies a given tax x. In this case, one can think of the shock as altering the implied deadweight loss associated with a tax x or how distortionary a tax x is in the economy. Paczos and Shakhnov (2018) show that a volatile tax wedge can be microfounded in different ways, for instance, in a model with endogenous labor supply and labor income tax.

[6]Since the game between the household and her alter ego is zero sum, the timing protocol between their moves do not affect the solution.

In the model economy, the government would want to promote utility by increasing the real money holdings towards the satiation level. In equilibrium, however, this can only be achieved by reducing the money supply over time, which in turn induces a gradual deflationary process along the way. In order to balance its budget the government has to set positive taxes withdrawing money from circulation. Taxes are assumed to be distortionary, and, hence, this has negative effects on households' income.

In this simple framework, as discussed by Calvo (1978) and Chang (1998), the optimal policies for the Ramsey government with the ability to commit would typically be time-inconsistent. A discussion of the source of the time-inconsistency of the *Ramsey plan* is presented in section 4.

3 Competitive Equilibrium with Model Uncertainty

In this section I define and characterize a competitive equilibrium with model uncertainty in this economy. Throughout the rest of the paper bold letters will be used to denote state-contingent sequences.

Definition 3.1. *A government policy in this economy is given by sequences of (inverse) money growth rates $\boldsymbol{h} = \{h_t(s^t)\}_{t=0}^{\infty}$ and tax collections $\boldsymbol{x} = \{x_t(s^t)\}_{t=0}^{\infty}$. A price system is $\boldsymbol{q} = \{q_t(s^t)\}_{t=0}^{\infty}$. An allocation is given by a triple of nonnegative sequences of consumption, real balances and income, $(\boldsymbol{c}, \boldsymbol{m}, \boldsymbol{y})$, where $\boldsymbol{c} = \{c_t(s^t)\}_{t=0}^{\infty}$, $\boldsymbol{m} = \{m_t(s^t)\}_{t=0}^{\infty}$, and $\boldsymbol{y} = \{y_t(s^t)\}_{t=0}^{\infty}$.*

Definition 3.2. *Given M_{-1}, s_0, a competitive equilibrium with model uncertainty is given by an allocation $(\boldsymbol{c}, \boldsymbol{m}, \boldsymbol{y})$, a price system $\boldsymbol{q}$, belief distortions $\boldsymbol{d}$, and a sequence of households' utility values $\mathbf{V^H} = \{V_{t+1}^H\}_{t=0}^{\infty}$ such that for all t and all s^t*

   (i) *given $\mathbf{q}$, beliefs' distortions $\mathbf{d}$, and government's policies $\mathbf{h}$ and $\mathbf{x}$, $(\mathbf{m}, \mathbf{V^H})$ solves households' maximization problem;*

  (ii) *given $\mathbf{q}$ and $(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{V^H})$, $\mathbf{d}$ solves the minimization problem;*

 (iii) *government's budget constraint holds;*

 (iv) *money and consumption good markets clear, i.e. $c_t(s^t) = y_t(s^t)$ and $m_t(s^t) = q_t(s^t)M_t(s^t)$.*

Under assumptions [A1-A6] I can prove the following proposition:

PROPOSITION 1. *A competitive equilibrium is completely characterized by sequences* $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right)$ *such that for all t and all s^t, $m_t\left(s^t\right) \in \mathbb{M}$, $x_t\left(s^t\right) \in \mathbb{X}$, $h_t\left(s^t\right) \in \Pi$, $d_{t+1}\left(s^{t+1}\right) \in \mathbb{D} \subseteq \mathbb{R}_+^S$, and $V_{t+1}^H(s^{t+1}) \in \mathbb{V}$ and*

$$m_t\left(s^t\right)\left\{u'(f(x_t\left(s^t\right), s_t)) - v'(m_t\left(s^t\right))\right\} =$$
$$\beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)d_{t+1}(s_{t+1}|s^t)\left\{u'(f(x_{t+1}\left(s^{t+1}\right), s_{t+1})h_{t+1}\left(s^{t+1}\right)m_{t+1}\left(s^{t+1}\right)\right\} \quad, \quad \leq \quad if \; m_t = \overline{m}$$

(9)

$$d_{t+1}(s_{t+1}|s^t) = \frac{\exp\left(-\frac{V_{t+1}^H\left(s^{t+1}\right)}{\theta}\right)}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\exp\left(-\frac{V_{t+1}^H(s^{t+1})}{\theta}\right)} \tag{10}$$

$$V_t^H = u\left(f(x_t\left(s^t\right), s_t)\right) + v\left(m_t\left(s^t\right)\right) - \beta\theta \log \sum_{s_{t+1}} \pi\left(s_{t+1}|s_t\right)\exp\left(\frac{-V_{t+1}^H\left(s^{t+1}\right)}{\theta}\right) \tag{11}$$

$$-x_t\left(s^t\right) = m_t\left(s^t\right)\left(1 - h_t\left(s^t\right)\right). \tag{12}$$

Proof. See Appendix A.1. □

Equation (9) is an Euler equation for real money balances. Equation (10) is simply the exponential twisting formula for optimal probability distortions, rewritten from (4). Equation (11), as in (2.1), expresses the household's utility values recursively once the probability distortions chosen by the evil alter ego are incorporated. Finally, equation (12) is the government's balanced budget constraint.

Note that households' transversality condition is not included in the list of conditions characterizing competitive equilibrium. Appendix A.1 explains why this is the case.

Formally, let $\mathbb{E} \equiv \mathbb{M} \times \mathbb{X} \times \Pi \times \mathbb{D} \times \mathbb{V}$ and $\mathbb{E}^\infty \equiv \mathbb{M}^\infty \times \mathbb{X}^\infty \times \Pi^\infty \times \mathbb{D}^\infty \times \mathbb{V}^\infty$. I define a set of competitive equilibria for each possible realization of the initial state s_0

$$CE_s = \left\{\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in \mathbb{E}^\infty \mid \; (9)\text{-}(12) \text{ hold and } s_0 = s\right\}.$$

Appendix A.2 presents an example of a competitive equilibrium sequence.

COROLLARY 1. CE_s *for all $s \in \mathbb{S}$ is nonempty.*

Proof. See Appendix A.2. □

COROLLARY 2. CE_s *for all* $s \in \mathbb{S}$ *is compact.*

Proof. See Appendix A.4. □

COROLLARY 3. *A continuation of a competitive equilibrium with model uncertainty is a competitive equilibrium with model uncertainty, i.e. if* $(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}) \in CE_{s_0}$ *then* $\left\{m_t, x_t, h_t, d_t, V_{t+1}^H\right\}_{j=t}^{\infty} \in CE_{s_t}$ *for all* t *and all* $s_0, s_t \in \mathbb{S}$.

Proof. Follows immediately from Proposition 1. □

4 Ramsey Problem for a Paternalistic Government: Towards a Recursive Formulation

I start by formulating and solving the government's Ramsey problem. Although the assumption that the government has the ability to commit might be put in question, studying such environment will be useful for two reasons. First, it will allow us to describe the notion of a paternalistic government and to characterize the set of equilibrium values (both for the government and households) that the government can attain in this economy with fragile beliefs with commitment. This set of equilibrium values is interesting as it constitutes a larger set which includes the set of values that could be delivered when the government chooses sequentially. The discrepancy between these two sets sheds some light on the severity of the time-inconsistency problem. Second, as it will become clearer later on, the procedure for solving the Ramsey problem will constitute a helpful step towards deriving a recursive structure for the credible plans.

I assume first that the government sets its policy once and for all at time zero. That is, at time zero it chooses the entire infinite sequence of money growth rates $\{h_t(s^t)\}_{t=0}^{\infty}$ and commits to it. A benevolent government in this economy would exhibit households' preference orderings and, hence, maximize households' expected utility under the distorted model given by (1). In the setup, I depart from the assumption of a benevolent government, and assume instead that the government is *paternalistic* in the sense that it cares about households' utility but under its own beliefs, which are assumed to be $\pi(s^t)$. The assumption of a paternalistic government implies in turn that the households and the government do not necessarily share the same beliefs when

evaluating contingent plans for consumption and real balances.[7] While the government believes that the exogenous shock evolves according to the approximating model $\pi(s^t)$, the households act as if the evolution of the shock is governed by $\widetilde{\pi}(s^t)$.

For a given initial shock realization s_0 and initial M_{-1}, the Ramsey problem that the government solves therefore consists of choosing $(m, x, h, d) \in CE_{s_0}$ to maximize households' expected utility under the approximating model, i.e.,

$$V_t^G = \max_{(\mathbf{m,x,h,d,V^H})} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) \left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right)) \right] \quad \text{s.t. (9) - (12).} \tag{13}$$

I solve the Ramsey problem by formulating it in a recursive fashion. To do so, I need to adopt a recursive structure for the competitive equilibria. It is key then to identify any variables that summarize all relevant information about future policies and future allocations for households' decisionmaking in the current period. From the Euler equation (9) one immediately identifies the variables needed. For time t, history s^t, and the households' choice of real balances $m_t(s^t)$, I need to know the (discounted) expected value of money at $t + 1$, defined by the right hand side of equation (9). The expected value of money at $t + 1$ can be expressed in terms of the value of money for each shock realization s_{t+1} and the probability distribution households assign to s_{t+1}. Following Kydland and Prescott (1980) and Chang (1998), I designate the value of money as a pseudo-state variable to track.[8] Let $\mu_{t+1}(s^{t+1})$ denote the equilibrium value of money at $t + 1$ after history s^{t+1},

$$\mu_{t+1}\left(s^{t+1}\right) \equiv u'(f(x_{t+1}\left(s^{t+1}\right), s_{t+1})(h_{t+1}\left(s^{t+1}\right) m_{t+1}\left(s^{t+1}\right)). \tag{14}$$

One can view $\mu_{t+1}(s^{t+1})$ as the "promised" (adjusted) marginal utility of money after s^{t+1}.

The second ingredient needed to compute the expected value of money at $t + 1$ is households' beliefs about s_{t+1}. As shown in Hansen and Sargent (2007), households

[7]This assumption seems to be standard in the incomplete information frameworks. See, for example, Cukierman and Meltzer (1986), Lorenzoni (2010). In frameworks where multiple probability distributions are contemplated, Woodford (2010) and Karantounias (2013) both feature paternalistic Ramsey planners. In the latter case, however, it is the government that is assumed to have multiplier preferences while the public uses a unique probability distribution.

[8]To solve for the Ramsey plan in a dynamic economy with capital accumulation, Marcet and Marimon (2009) instead use the Lagrange multiplier associated with the Euler equations as a pseudo-state variable to guarantee that they are satisfied at every point of time.

want to guard themselves against a worst-case scenario by twisting the approximating probability model in accordance to distortions $d_{t+1}(s^{t+1})$. Therefore, the future paths of $h_{t+1}(s^{t+1})$ and $m_{t+1}(s^{t+1})$ influence today's choice of real money balances m_t, not only through their effect on $\mu_{t+1}(s^{t+1})$ but also through the impact they have on the degree of distortion in the beliefs of the representative household, as given by (10).

These probability distortions in equilibrium turn out to be a function of households' continuation values. Therefore, to construct a recursive representation of the competitive equilibria with model uncertainty one needs to compute households' utility values $V^H(s^{t+1})$, in addition to $\mu_{t+1}(s^{t+1})$. Together, these can be thought of as device used to ensure that we account for the effects of future policies on agents' behavior in earlier periods.

Let $\Re^2$ be the space of all the subsets of $\mathbb{R}^2$. Moreover, let $\Omega : \mathbb{S} \to \Re^2$ be the value correspondence such that

$$\Omega \left(s = s_0\right) = \Big\{ \left(\mu_s, V_s^H\right) \in \mathbb{R} \times \mathbb{R} |\ \mu_s \equiv u' \left[f(x_0\left(s_0\right), s_0)\right] \left[x_0\left(s_0\right) + m_0\left(s_0\right)\right] \text{ and}$$

$$V_s^H = u \left(f(x\left(s_0\right), s_0) + v \left(m\left(s_0\right)\right) - \beta\theta \log \sum_{s_1} \pi \left(s_1|s_0\right) \exp \left(\frac{-V_1^H(s_1)}{\theta}\right)\right.$$

$$\text{with } s_0 = s \text{ and for some } (\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}) \in CE_s \Big\}.$$

For each initial state realization s, the set $\Omega(s)$ is formed by all current (adjusted) marginal utilities and households' values that can be delivered in a competitive equilibrium. Through these two variables, future policies and allocations $(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H})$ influence the choice of m_0 for $s_0 = s$. It is straightforward to check that $\Omega(s)$ is nonempty and compact.

Define

$$\Psi \left(s, \mu_s, V_s^H\right) = \Big\{ (\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}) \in CE_s | \mu_s = u' \left[f(x_0\left(s_0\right), s_0)\right] \left[x_0\left(s_0\right) + m_0\left(s_0\right)\right]$$

and

$$V_s^H = u \left(f(x\left(s_0\right), s_0) + v \left(m\left(s_0\right)\right) - \beta\theta \log \sum_{s_1} \pi \left(s_1|s_0\right) \exp \left(-V_1^H \left(s_1\right)/\theta\right) \Big\}.$$

$\Psi \left(s, \mu_s, V_s^H\right)$ delivers the competitive equilibrium sequences $(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H})$ associated with an initial marginal utility μ_s and an initial lifetime utility for the households V_s^H for initial $s_0 = s$. If we know sets $\Omega(s)$ and $\Psi \left(s, \mu_s, V_s^H\right)$, we could solve the Ramsey problem for the paternalistic government in (13) for $s_0 = s$ in two steps as follows. First, I solve the Ramsey problem when the current shock realization is s and

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the current marginal utility and households' value are μ_s and V_s^H, respectively,

$$V^{G*}(s, \mu_s, V_s^H) = \max_{(\mathbf{m,x,h,d,V^H})} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) \left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right)) \right]. \tag{15}$$

$$s.t. \quad \left(\mathbf{m, x, h, d, V^H}\right) \in \Psi\left(s, \mu_s, V_s^H\right)$$

Let $\mu = [\mu_1, \mu_2, ..., \mu_S]$ and $V^H = \left[V_1^H, V_2^H, ..., V_S^H\right]$ be the vectors of state-contingent marginal utilities and households' utilities, respectively. Notice that $\mu_s \in [0, \overline{\mu_s}]$ for some $\overline{\mu_s}$, $\forall s \in \mathbb{S}$. Also, given that the period payoffs are bounded, it follows that $V_s^H \in \left[\underline{V}_s^H, \overline{V}_s^H\right]$, for some bounds $\underline{V}_s^H, \overline{V}_s^H$. The primes are used to denote next-period values.

The next proposition formulates the Ramsey problem (15) with a recursive structure that can be solved using dynamic programming techniques.

PROPOSITION 2. $V^{G*}\left(s, \mu_s, V_s^H\right)$ satisfies the following Bellman equation

$$V^G\left(s, \mu_s, V_s^H\right) = \max_{(m,x,h,\mu',V^{H\prime})} \left[u\left(f\left(x,s\right)\right) + v(m)\right] + \beta \sum_{s'} \pi(s'|s) w_{s'}\left(s', \mu'_{s'}, V_{s'}^{H\prime}\right) \tag{16}$$

$$(m, x, h) \in \mathbb{M} \times \mathbb{X} \times \Pi \text{ and } \left(\mu'_{s'}, V_{s'}^{H\prime}\right) \in \Omega\left(s'\right) \text{ for all } s'$$

$$\mu_s = u'\left[f(x,s)\right]\left[x + m\right] \tag{17}$$

$$V_s^H = u\left(f(x,s)\right) + v\left(m\right) - \beta\theta \log \sum_{s'} \pi\left(s'|s\right) \exp\left(\frac{-V_{s'}^{H\prime}}{\theta}\right) \tag{18}$$

$$-x = m\left[1 - h\right] \tag{19}$$

$$m\left\{u'(f(x,s)) - v'(m)\right\} = \beta \sum_{s'} \pi(s'|s) \frac{\exp\left(-\frac{V_{s'}^{H\prime}}{\theta}\right)}{\sum_{s'} \pi(s'|s) \exp\left(-\frac{V_{s'}^{H\prime}}{\theta}\right)} \mu'_{s'}, \quad \leq \text{ if } m = \overline{m}. \tag{20}$$

Conversely, if a bounded function $V^G : \mathbb{S} \times \Omega(s) \to \mathbb{R}$ satisfies the above Bellman equation, then it is solution of (15).

Proof. Based on the Bellman principle of optimality, this is a straightforward extension of Chang (1998), p. 457, and is left to the reader. $\square$

In the recursive Ramsey problem given by (16) it is clear to see how when maximizing its utility in any period $t > 0$ the government is bounded by its previous-period promises of marginal utility and households' value (μ, V^H). From the households' perspective, these promises were key when choosing real balances at $t-1$. To maximize their utility,

the time $t-1$ Euler equation has to hold. Under commitment, these promises must be delivered at t thereby conditioning government's choice in that period. In this way, the government guarantees that households' Euler equation is satisfied in every period. Through the dynamics of the promised marginal utility and households' value, which the government has to manage to deliver in equilibrium, the Ramsey plan exhibits history dependence. Once we have solved the recursive Ramsey problem, the following step has to be undertaken

$$V^{G*}(s) = \max_{(\mu_s, V_s^H) \in \Omega(s)} V^{G*}\left(s, \mu_s, V_s^H\right). \tag{21}$$

In contrast with the other periods, there is no promised (μ_s, V_s^H) to be delivered in the first period. Hence, the government is free to choose the initial vector (μ_s, V_s^H).[9] Consider first the choice of the Ramsey planner at $t = 0$. The following Theorem 1 characterizes that choice and compares it with the choice of the Ramsey planner in an economy where a representative household is assumed to have rational expectations, as opposed to the fragile beliefs considered here (I will denote the variables in such a rational-expectations counterpart economy with superscripts RE).

THEOREM 1. *Consider two economies: one in which the planner's and households' beliefs coincide (and are summarized by $\pi_t(s_{t+1}|s_t)$ for all t and all s_t) and another one - presented here - with households who hold fragile beliefs. Then, $V^G > V^{G,RE}$ where $V^{G,RE}$ defines the Ramsey value in the economy with rational expectations.*

Proof. See Appendix A.4. □

The proof of the above Theorem relies on two steps. First, I prove that the necessary condition for optimality of policy at $t = 0$ entails setting $f'(x_0^*, s_0) = 0$ in both economies, i.e. a stationary policy with zero taxes at time 0 which can be achieved with a zero rate of money growth, $h_0^*(s_0) = 1$. But in the economy with fragile beliefs that constant rate of money growth must be supported by higher real money holdings, i.e. $m_0^*(s_0) > m_0^{*,RE}(s_0)$ which results in higher welfare value. The reason why households with fragile beliefs demand higher real money balances has to do with a precautionary savings motive induced by the presence of aversion to model misspecification. What

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[9]The fact that (μ_s, V_s^H) can be set by the government at time 0 explains why I refer to it as *pseudo*-state variables.

happens beyond $t = 0$? In both economies, it is desirable to bring the quantity of money towards the satiation level. Starting with a higher level, the planner in an economy populated by households with fragile beliefs can afford reducing the supply of money steadily, at a slower pace than the planner in the RE-counterpart economy, inducing less tax distortion along the equilibrium deflation path, which results in higher lifetime welfare.

To fully solve the recursive problem stated in Proposition 2, it is necessary to know in advance the value correspondence Ω. In what follows I provide a procedure for the computation of Ω as the largest fixed point of a specific value correspondence operator, as proposed by Kydland and Prescott (1980).

Let $\mathcal{G}$ be the space of all the correspondences Ω, and let Q live in it. Let the operator $B : \mathcal{G} \to \mathcal{G}$ be defined as follows,

$$B\left(Q\right)(s) = \left\{\left(\mu_s, V_s^H\right) \in \mathbb{R} \times \mathbb{R} \mid \exists \left(m, x, h, \mu', V^{H\prime}\right) \in \mathbb{M} \times \mathbb{X} \times \Pi \times Q \text{ such that}\right.$$
$$\left.(17)\text{-}(20) \;\; \text{hold}\right\}.$$

By picking vectors of continuation marginal utilities and households' values $(\mu', V^{H\prime})$ from Q, the operator B computes the set of current marginal utilities and households' values (μ_s, V_s^H) for each shock realization s that are consistent with the competitive equilibrium conditions. The operator B is a monotone operator in the sense that $Q(s) \subseteq Q'(s)$ implies $B(Q)(s) \subseteq B(Q')(s)$.

The next proposition states that the set in question, $\Omega(s)$, is the largest fixed point of the operator B. Moreover, it states that $\Omega(s)$ can be computed by iterating on the operator B until convergence, given that I start from an initial set $Q_0(s)$ that is sufficiently large.

Let $Q_0(s) = [0, \overline{\mu_s}] \times \left[\underline{V}_s^H, \overline{V}_s^H\right]$. Clearly, it satisfies $B(Q_0)(s) \subseteq Q_0(s)$. Given the monotonicity property, by applying successively the operator B, we can construct a decreasing sequence $\{Q_n(s)\}_{t=0}^\infty$ for each $s \in \mathbb{S}$, where $Q_n(s) = B\left(Q_{n-1}\right)(s)$. The limiting sets are given by $Q_\infty(s) = \cap_{n=0}^\infty Q_n(s)$ for $n = 1, 2, ...$

PROPOSITION 3.

(i) $Q(s) \subseteq B\left(Q\right)(s) \Rightarrow B\left(Q\right)(s) \subseteq \Omega(s)$

(ii) $\Omega(s) = B\left(\Omega\right)(s)$

(iii) $\Omega(s) = \lim_{n \to \infty} B^{\infty}(Q_0)(s)$.

Proof. Simple extension of the argument in Chang (1998). □

Once we have computed Ω, we can solve the recursive Ramsey problem stated in Proposition (2) which clearly yields a Ramsey plan with a recursive representation. The resulting Ramsey plan consists of an initial vector (μ_s, V_s^H), given by the solution to (21), and a five-tuple of functions (h, x, m, μ, V^H) mapping (s, μ_s, V_s^H) into current period's (h, x, m), and next period's state-contingent (μ, V^H), respectively,

$$
\begin{aligned}
h_t &= h\left(s_t, \mu_t(s_t), V_t^H(s_t)\right) \\
x_t &= x\left(s_t, \mu_t(s_t), V_t^H(s_t)\right) \\
m_t &= m\left(s_t, \mu_t(s_t), V_t^H(s_t)\right) \\
\mu_{t+1} &= \psi\left(s_t, \mu_t(s_t), V_t^H(s_t)\right) \\
V_{t+1}^H &= \varpi\left(s_t, \mu_t(s_t), V_t^H(s_t)\right).
\end{aligned}
$$

As it turns out, the solution to the Ramsey problem is time-inconsistent. In this environment, the government would implement a transitory deflationary process along with a contracting money supply $\{M_t(s^t)\}_{t=0}^{\infty}$ so as to increase the real money holdings towards its satiation level, $\overline{m}$. To achieve this, it would have to collect tax revenues to satisfy its balanced budget constraint (8), which at the same time would entail tax distortions in the form of output costs. At the beginning of time zero, taking prices $\{q_t(s^t)\}_{t=0}^{\infty}$ and taxes $\{x_t(s^t)\}_{t=0}^{\infty}$ as given, the household chooses once and for all her sequence of real balances $\{m_t(s^t)\}_{t=0}^{\infty}$. If the government was allowed to revisit its policy at time $T > 0$, after history s^t, given households' choice $\{m_t(s^t)\}_{t=0}^{\infty}$, the government would find it optimal not to adhere to what the original plan prescribes from then on, $\{M_t(s^t|s^T)\}_{t=T}^{\infty}$, but to deviate to an alternative $\{\widetilde{M}_t(s^t|s^T)\}_{t=T}^{\infty}$ by reducing the money supply more gradually. These incentives arise from the fact that tax distortions are an increasing and convex function of tax collections, as indicated in assumption [A6].

5 Sustainable Plans with Model Uncertainty

From now on, I proceed under the assumption that the government cannot commit to its announced sequence of money supply growth rates. Instead, it will be choosing its

policy actions sequentially in each state.[10]

As originally studied by Calvo (1978) and explained in section 4, in this case the government faces a credibility problem. To study the optimal credible policies in this context, we make use of the notion of *sustainable plans*, developed by Chari and Kehoe (1990). The notion of a sustainable plan inherits sequential rationality on the government's side, combined with the fact that households are restricted to choose from competitive equilibrium allocations.[11]

In this section, I extend the notion of sustainable plans of Chari and Kehoe (1990) to incorporate model uncertainty.

Let $h^t = (h_0, h_1, ..., h_t)$ be the history of the (inverse) money growth rates in all the periods up to t. A *strategy for the government* can be defined as $\sigma^G \equiv \{\sigma_t^G\}_{t=0}^\infty$, with $\sigma_0^G : \mathbb{S} \to \Pi$ and $\sigma_t^G : \mathbb{S}^t \times \Pi^{t-1} \to \Pi$ for all $t > 0$. I restrict the government to choose a strategy σ^G from the set CE_s^Π, where CE_s^Π is defined as

$$CE_s^\Pi = \left\{h \in \Pi^\infty | \text{ there is some } \left(\mathbf{m}, \mathbf{x}, \mathbf{d}, \mathbf{V^H}\right) \text{ such that } \left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in CE_s\right\}.$$

CE_s^Π is the set of sequences of money growth rates consistent with the existence of competitive equilibria, given $s_0 = s$. It is straightforward to establish that this set is nonempty and compact.

The restriction above is equivalent to forcing the government to choose after any history h^{t-1}, s^t a period t money supply growth rate from $CE_{s_t}^{\Pi,0}$, where $CE_s^{\Pi,0}$ is given by

$$CE_s^{\Pi,0} = \{h \in \Pi : \text{ there is } \mathbf{h} \in CE_s^\Pi \text{ with } h(0) = h\}.$$

An *allocation rule* can be defined as $\alpha \equiv \{\alpha_t\}_{t=0}^\infty$ such that $\alpha_t : \mathbb{S}^t \times \Pi^t \to \mathbb{M} \times \mathbb{D} \times \mathbb{X}$ for all $t \geq 0$. The allocation rule α assigns an action vector $\alpha_t(s^t, h^t) = (m_t, x_t, d_t)(s^t, h^t)$ for current real balances, tax collections, and distortions to households' beliefs about the next state s_{t+1}.

---

[10]We can think instead of this environment as having a sequence of government "administrations" with the time t, history s^t administration choosing only at time t, history s^t government action given its forecasts of how future administrations will act. The time t, history s^t administration intends to maximize the government's lifetime utility only in that particular node.

[11]From a game theoretical perspective, the notion of a sustainable plan entails subgame perfection in a game between a large player (government) and a continuum of atomistic players (households), who cannot coordinate, and are, thus, price-takers.

Definition 5.1. *A government strategy, σ^G, and an allocation rule α, are said to constitute a* sustainable plan with model uncertainty *(SP) if after any history s^t and h^{t-1}*

(i) *(σ^G, α) induce a competitive equilibrium sequence;*

(ii) *given σ^H, it is optimal for the government to follow the continuation of σ^G , i.e. the sequence of continuation future induced by σ^G maximizes*

$$\sum_{j=t}^{\infty} \beta^{j-t} \sum_{s^j|s^t} \pi_j(s^j|s^t) \left[u\left(c_j\left(s^j\right)\right) + v(m_j\left(s^j\right)) \right] \quad \text{over the set } CE_s^{\Pi}.$$

Condition (i) states that after any history s^t, h^t, even if at some point in the past the government has disappointed households' expectations about money growth rates, all economic agents choose actions consistent with a competitive equilibrium. Condition (ii) guarantees that the government attains weakly higher lifetime utility after any history by adhering to σ^G.

Any sustainable plan with model uncertainty (σ^G, α) can be factorized after any history into a current period action profile, a, and a vector $(V^{G\prime}(h), V^{H\prime}(h), \mu'(h))$ of state-contingent continuation values for the government, and for the representative household, and promised marginal utilities, as a function of money growth rate h. The action profile a in this context is given by $a = (\widehat{h}, m(h), x(h), d'(h))$. That is, the action profile a assigns:

(i) an (inverse) money growth rate $\widehat{h}$ that the government is instructed to follow

(ii) a reaction function $m : \Pi \to [0, \overline{m}]$ for the real money holdings chosen by households. If the government adheres to the plan and executes recommended $\widehat{h}$, households respond by acquiring $m(\widehat{h})$ real balances. Otherwise, if the government deviates from the sustainable plan and select any $h \neq \widehat{h}$, households react by selecting $m(h)$.

(iii) a tax allocation rule $x : \Pi \to \mathbb{X}$. Taxes revenues are determined in equilibrium as a residual of money growth and money holdings in order to satisfy the government's budget constraint (7).

(iv) a reaction function $d : \Pi \to \mathbb{D}$ for the beliefs' distortions set by the evil alter ego.

The vector $(V^{G\prime}(h), V^{H\prime}(h), \mu'(h))$ reflects how continuation outcomes are affected by the current choice h of the government through the effect it has on households' expectations and thereby on future prices. Given the timing protocol within the period, households' response or punishment to a government deviation $h \neq \widehat{h}$ consists of an action $m(h)$, typically different from $m(\widehat{h})$, in the same period, followed by subsequent actions and associated future equilibrium prices, the impact of which is captured by $(V^{G\prime}(h), V^{H\prime}(h), \mu'(h))$.

In this context, the sustainable plans combine two sources of history dependence. In addition to the one embedded in the dynamics of the marginal utilities, as in the Ramsey plan, there is a new source of history dependence arising from the restrictions that a system of households' expectations impose on the government's policy actions. As the government after any history is allowed to revisit its announced policy and reset it, households expect that the government will adhere to the original plan only if it is in its own interest to do so.

Let $\mathbb{A}(s)$ be given by

$$\mathbb{A}(s) = \left\{ \left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in CE_s \mid \text{there is a SP whose outcome is } \left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \right\}.$$

Let $\Re^3$ be the space of all the subsets of $\mathbb{R}^3$. I define the value correspondence $\Lambda : \mathbb{S} \longrightarrow \mathcal{R}^3$ as

$$\Lambda(s) = \left\{ \left(V_s^G, V_s^H, \mu_s\right) \mid \text{there is a } \left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in \mathbb{A}(s) \text{ with} \right.$$

$$V_s^G = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) \left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right)) \right],$$

$$V_s^H = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) D_t(s^t) \left\{ \left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right)) \right] \right.$$

$$\left. + \theta\beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) \log d_{t+1}(s_{t+1}|s^t) \right\},$$

$$\left. \mu_s = u'\left[f(x_0\left(s_0\right), s_0)\right]\left[x_0\left(s_0\right) + m_0\left(s_0\right)\right] \right\}.$$

For each $s \in \mathbb{S}$, $\Lambda(s)$ constitutes the set of vectors of equilibrium values for the government and the household, and the promised marginal utilities given state s that can be delivered by a sustainable plan. I denote as $\widehat{\mathcal{G}}$ the space of all such correspondences.

Definition 5.2. *For any correspondence $Z \subset \widehat{\mathcal{G}}$, $(a, V^{G\prime}(\cdot), V^{H\prime}(\cdot), \mu'(\cdot))$ is said to be admissible with respect to Z at state s if*

(i) $a = (\widehat{h}, m(h), x(h), d'(h)) \in \Pi \times [0, \overline{m}]^{\Pi} \times X^{\Pi} \times \mathbb{R}^{\Pi}$;

(ii) $(V_{s'}^{G'}(h), V_{s'}^{H'}(h), \mu_{s'}'(h)) \in Z(s') \qquad \forall\, h \in CE_s^{\Pi,0}, s' \in \mathbb{S}$;

(iii) (19)-(20) are satisfied;

(iv) $u(f(x(\widehat{h}), s)) + v(m(\widehat{h})) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V_{s'}^{G'}(\widehat{h}) \geq$
$$u(f(x(h), s)) + v(m(h)) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V_{s'}^{G'}(h) \qquad \forall h \in CE_s^{\Pi,0}.$$

Condition (i) ensures that a belongs to the appropriate action space. Condition (ii) guarantees that for any h that the government contemplates the vector of continuation values and promised marginal utility for next period's shock s' belongs to the corresponding set $Z(s')$. Condition (iii) imposes the competitive equilibrium conditions in the current period. Finally, condition (iv) describes the incentive constraint for the government in the current period. This incentive constraint deters the government from taking one-period deviations when contemplating money growth rates h other than prescribed $\widehat{h}$. If condition (iv) holds, it follows from the "one-period deviation principle" that there are no profitable deviations at all. A plan is credible if the government finds it is in its own interest to confirm households' expectations about its policy action $\widehat{h}$. Condition (iv) guarantees that this is the case.

In what follows, I explain how to compute the equilibrium value sets $\Lambda(s)$. Let $Z \subset \widehat{\mathcal{G}}$. In the spirit of Abreu, Pearce, and Stacchetti (1990) I construct the operator $\widehat{B} : \widehat{\mathcal{G}} \longrightarrow \widehat{\mathcal{G}}$ as follows

$$\widehat{B}(Z)(s) = co\Big\{(V_s^G, V_s^H, \mu_s)| \exists \text{ admissible } (a, V^{G'}(\cdot), V^{H'}(\cdot), \mu'(\cdot)) \text{ with respect to } Z \text{ at } s:$$
$$a = (\widehat{h}, m(h), x(h), d'(h))$$
$$V_s^G = u(f(x(\widehat{h}), s)) + v(m(\widehat{h})) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V_{s'}^{G'}(\widehat{h})$$
$$V_s^H = u(f(x(\widehat{h}), s)) + v(m(\widehat{h})) - \beta\theta \log \sum_{s' \in \mathbb{S}} \pi(s'|s) \exp\left\{-\frac{V_{s'}^{H'}(\widehat{h})}{\theta}\right\}$$
$$\mu_s = u(f(x(\widehat{h}), s))(x(\widehat{h}) + m(\widehat{h}))\Big\}.$$

For each $s \in \mathbb{S}$, $\widehat{B}(Z)(s)$ is the convex hull of the set of vectors (V_s^G, V_s^H, μ_s) that can be sustained by some admissible action profile a and vectors $(V_s^{G'}, V_s^{H'}, \mu_s')$ of continuation values and marginal utilities in $Z(s')$ for each state s' next period.

I assume that there exists a public randomization device.[12] In particular, I assume that every period an exogenous, serially uncorrelated public signal $\widetilde{X}_t$ is drawn from a $[0,1]$ uniform distribution. Depending on current actions, this signal will determine which equilibrium will be played next period. While the public is assumed to contemplate multiple probability distributions governing the evolution of the s shock, they know and fully trust the distribution for this public signal.[13]

The following propositions are simple adaptations of Abreu, Pearce, and Stacchetti (1990) for repeated games and establish some useful properties of the operator. Together, these properties guarantee that the equilibrium value correspondence Λ is its largest fixed point and can be found by iterating on this operator.

PROPOSITION 4. *Monotonicity:* $Z \subseteq Z'$ *implies* $B(Z) \subseteq B(Z')$.

Proof. The proof is a simple extension of that in Chang (1998). □

PROPOSITION 5. *Self-Generation: If $Z(s)$ is bounded and $Z(s) \subseteq B(Z)(s)$, then $B(Z)(s) \subseteq \Lambda(s)$.*

Proof. We need to construct a subgame perfect strategy profile (σ^G, σ^H) such that

(i) for each $s \in \mathbb{S}$ it delivers a lifetime utility value V_s^G to the government, V_s^H to a representative household with an associated marginal promised utility μ_s,

(ii) the associated outcome of the SP satisfies (19)-(20)

(iii) government's incentive constraint holds for every history (s^t, h^{t-1}).

To do so, fix an initial state s and consider any $(V_s^G, V_s^H, \mu_s) \in B(Z)(s)$. Let $(V_0^G, V_0^H, \mu_0) = (V_s^G, V_s^H, \mu_s)$ and define (σ^G, σ^H) recursively as follows.

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[12]For other application of public randomization device see, for example, Phelan and Stacchetti (2001).

[13]This assumption is akin to stating that some sources of randomness might be better understood than others, like roulette wheels versus horse races (Anscombe and Aumann (1963)). In the robust control literature such a distinction is allowed for by the construction with the so-called T1 and T2 operators corresponding to different degrees of aversion towards uncertainty with respect to different fundamental shocks (see, for example, Boyarchenko (2012), Pouzo and Presno (2016)). For the public randomization device to be correctly applied in this set-up, it is further important to time it right so that agents are not able to hedge against model uncertainty / ambiguity by randomizing (see Wolitzky; I am grateful to Tomasz Strzalecki for pointing me to this reference).

Let $(V_t^G(h^{t-1}, s^{t-1}, s_t), V_t^H(h^{t-1}, s^{t-1}, s_t), \mu_t(h^{t-1}, s^{t-1}, s_t)) \in Z(s_t)$ be the vector of values and marginal utilities after an arbitrary history (h^{t-1}, s^{t-1}, s_t). Since $Z \subset B(Z)$, for each $s \in \mathbb{S}$ there exists an admissible vector $(\widehat{h}, m(h), x(h), d'(h), V^{G'}(h), V^{H'}(h), \mu'(h))$ with respect to Z at s. Define $\sigma_t^G(h^{t-1}, (s^{t-1}, s_t)) = \widehat{h}$ and $\widehat{m} = m(h)$. Let $\alpha_t(h^{t-1}, (s^{t-1}, s_t)) = (m(h), m(h)(h-1), d'(h))$ if $h \in CE_{s_t}^{\Pi,0}$ and $= (0, 0, d'^{NM})$ otherwise, where d'^{NM} are the probability distortions corresponding to the nonmonetary equilibrium.[14]

Also, define $(V_{t+1}^G(h^t, s^t, s_{t+1}), V_{t+1}^H(h^t, s^t, s_{t+1}), \mu_{t+1}(h^t, s^t, s_{t+1})) = (V_{s_{t+1}}^{G'}(h), V_{s_{t+1}}^{H'}(h), \mu'_{s_{t+1}}(h))$ if $h \in CE_{s_{t+1}}^{\Pi,0}$; $(V_{t+1}^G(h^t, s^t, s_{t+1}), V_{t+1}^H(h^t, s^t, s_{t+1}), \mu_{t+1}(h^t, s^t, s_{t+1})) = (V_{s_{t+1}}^{GNM}, V_{s_{t+1}}^{HNM}, \mu_{s_{t+1}}^{NM})$ otherwise. Clearly, $(V_{t+1}^G(h^t, s^t, s_{t+1}), V_{t+1}^H(h^t, s^t, s_{t+1}), \mu_{t+1}(h^t, s^t, s_{t+1})) \in Z(s_{t+1})$. By admissibility, (σ^G, α) is unimprovable and, thus, is subgame perfect. Since $Z(s)$ is bounded for every $s \in \mathbb{S}$, it is straightforward to show that (σ^G, α) delivers (V_s^G, V_s^H, μ_s). Also, admissibility of vectors $(\widehat{h}, m(h), x(h), d'(h), V^{G'}(h), V^{H'}(h), \mu(h))$ ensures that the equilibrium conditions are satisfied along the equilibrium path. $\qquad\square$

PROPOSITION 6. *Factorization:* $\Lambda = B(\Lambda)$.

Proof. By the previous proposition, it is sufficient to show that $\Lambda(s)$ is bounded and that $\Lambda(s) \subset B(\Lambda)(s)$. The latter result follows from the fact that the continuation of a sustainable plan is also a sustainable plan. The boundness of $\Lambda(s)$ follows from (i) the fact that any lifetime utility for the government is the expected discounted sum of one-period bounded payoffs; (ii) any lifetime utility for the household can be bounded by discounted sums of non-stochastic extremal one-period payoffs, and (iii) marginal utilities are determined by continuous functions f, u' over compact sets. $\qquad\square$

PROPOSITION 7. *If $Z(s)$ is compact for each $s \in \mathbb{S}$, then so is $B(Z)(s)$.*

Proof. Let us show first that $B(Z)(s)$ is bounded. Let $\overline{Z}$ be a value correspondence in $\widehat{\mathcal{G}}$. Define the operators $\Upsilon_{i,s} : \widehat{\mathcal{G}} \longrightarrow \mathcal{R}$ for $i = 1, 2$, where $\Re$ is the space of subsets in $\mathbb{R}$,

$$\Upsilon_{1,s}(\overline{Z}) = \{V_s^G : \exists (V_s^G, V_s^H, \mu_s) \in \overline{Z}(s)\}$$
$$\Upsilon_{2,s}(\overline{Z}) = \{V_s^H : \exists (V_s^G, V_s^H, \mu_s) \in \overline{Z}(s)\}.$$

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[14]Even though the continuation outcome in case the government selects h not belonging to $CE_{s_t}^{\Pi,0}$ is irrelevant for the solution (since it cannot occur by assumption), to be rigorous we need to specify the moves after any history. If the government executes h not in $CE_{s_t}^0$ I assume that the economy switches to the nonmonetary equilibrium.

Boundness of $B(Z)(s)$ follows from having

$$\Upsilon_{1,s}(B(Z)) \subset U_s^0 + \beta \sum_{s'} \pi(s'|s)\Upsilon_{1,s'}(Z)$$

$$\Upsilon_{2,s}(B(Z)) \subset U_s^0 - \beta\theta \log \sum_{s'} \pi(s'|s) \exp\left(-\Upsilon_{2,s'}(Z)/\theta\right),$$

where the sets of one-period payoffs U_s^0 (for current state s), and $\Upsilon_{i,s'}(Z)$ for $i = 1, 2$ are bounded.

Let us show now that $B(Z)(s)$ is closed. Consider any sequence $\left\{(V^{Gn}, V^{Hn}, \mu^n)\right\}_{n=1}^{+\infty}$ such that $(V_t^{Gn}(s^{t-1}, s_t), V_t^{Hn}(s^{t-1}, s_t), \mu_t^n(s^{t-1}, s_t)) \in B(Z)(s_t) \quad \forall s^{t-1} \in \mathbb{S}^{t-1}, s_t \in \mathbb{S}$ that converges to some (V^{G*}, V^{H*}, μ^*). Fix an arbitrary sequence of states $\{s_t\}_{t=0}^{+\infty}$. We need to show that

$$(V^{G*}(s^{t-1}, s_t), V^{H*}(s^{t-1}, s_t), \mu^*(s^{t-1}, s_t)) \in B(Z)(s_t) \; \forall s^{t-1} \in \mathbb{S}^t, s_t \in \mathbb{S}.$$

For each $(V_t^{Gn}(s^{t-1}, s_t), V_t^{Hn}(s^{t-1}, s_t), \mu_t^n(s^{t-1}, s_t))$, there exists an admissible vector $(\widehat{h}^n, m^n(h), x^n(h), d'^n(h), V^{Gn\prime}(h), V^{Hn\prime}(h), \mu^{n\prime}(h))$ with respect to Z at s. This vector should be indexed by histories of shocks s^t. In particular, $\widehat{h}_t^n(s^t) = \widehat{h}^n$. Since $\{s_t\}_{t=0}^{+\infty}$ is fixed, I slightly abuse the notation and refer to $\widehat{h}_t^n(s^t)$ as just $\widehat{h}_t^n$. Without loss of generality, I assume that $\widehat{h}_t^n$ converges to some $\widehat{h}_t^* \in CE_{s_t}^{\Pi,0}$. In a similar way, for each $h \in CE_{s_t}^{\Pi,0}$, $(m^n(h), x^n(h), d'^n(h), V^{Gn\prime}(h), V^{Hn\prime}(h), \mu^{n\prime}(h)) \longrightarrow (m^*(h), x^*(h), d'^*(h), V^{G\prime}(h)^*, V^{H\prime}(h)^*,$
$\mu'(h)^*)$ where $(m^*(h), x^*(h), d'^*(h)) \in [0, \overline{m}] \times \mathbb{X} \times \mathbb{D}$ and $(V_{s'}^{G\prime}(h)^*, V_{s'}^{H\prime}(h)^*, \mu_{s'}'(h)^*) \in Z(s') \; \forall s' \in \mathbb{S}$, by compactness of $[0, \overline{m}] \times \mathbb{X} \times \mathbb{D}$ and $Z(s') \; \forall s' \in \mathbb{S}$. By the continuity of functions u, v, f, u', v', it is straightforward to check that $(m^*(h), x^*(h), d'^*(h), V^{G\prime}(h)^*, V^{H\prime}(h)^*, \mu'(h)^*)$ satisfies conditions (19)-(20). It follows then that $(V^{G*}(s^{t-1}, s_t), V^{H*}(s^{t-1}, s_t), \mu^*(s^{t-1}, s_t)) \in B(Z)(s_t)$. $\qquad\square$

Orlik and Presno (2013) proceeded to implement the operator $\widehat{B}$ on the computer in order to compute the equilibrium value correspondence Λ. The computational algorithm (see Appendix A.5) is based on an outer approximation of the value sets and is a straightforward adaptation of the approach developed by Judd, Yeltekin, and Conklin (2003).[15]

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[15]Several techniques have been applied to find the equilibrium value sets in different environments. Chang (1998) uses an approach based on the discretization of both the space of actions and the space

6 Three-Period Model

The above analysis and the numerical analysis in Orlik and Presno (2013) entails characterizing the full set of sustainable equilibria in terms of continuation values and (distorted) marginal utilities associated with some underlying competitive equilibria. The analysis does not provide the characterization of the actual strategies, allocations and prices that support these equilibrium values. To shed further light on the dynamics of the model, and, in particular, on optimal government policies in the presence of uncertainty averse households I develop here a simplified model with three periods, $t = 0, 1, 2$. I start the argument by developing a rational-expectations version of this three-period model as it can be solved analytically. I assume that households are risk-neutral with respect to consumption risk. The only source of randomness in this economy is an i.i.d. shock that realizes in the period $t = 1$ and affects the extent to which a tax x reduces income. The shock can only take two values $s = 1, 2$ with probabilities $p(s)$, respectively. I consider the following functional forms

$$u(c) = c$$
$$v(m) = \psi(\bar{m} - 0.5m^2)$$
$$f(x, s) = \kappa - \phi_s x^2$$
$$g(x) = \kappa - \phi x^2$$

where $f(x, s)$ is the income function in period $t = 1$ given tax x and shock s, and $g(x)$ is the income function in periods $t = 0$ and $t = 2$, with $0 < \phi_1 < \phi_2$ and $\phi = \frac{\phi_1 + \phi_2}{2}$. Even in this economy with consumption-risk neutral households, the key trade-off for government policy remains endangering time-consistency of its policies. Inducing a deflationary process increases welfare by raising real money balances but, on the other hand, it also pushes up expected tax costs as more tax collections are required to retire money from circulation in order to reach the satiation level of real money holdings.

The Ramsey problem in this economy when the planner and household have common

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of continuation values and promised marginal utilities. This technique suffers from a severe curse of dimensionality. Instead, the method proposed by Judd, Yeltekin, and Conklin (2003) discretizes only the action space and by solving optimization problems approximates the value sets in question using hyperplanes. Also see Fernández-Villaverde and Tsyvinski (2002) for an adaptation of this procedure to characterize the value sets in a dynamic capital taxation model without commitment.

belief about the evolution of shock s can be written as

$$\max_{(m_0,m_1(s),m_2(s),h_0,h_1(s),h_2(s))} g\left[m_0(h_0 - 1)\right] + v(m_0)$$

$$+\beta \sum_s p(s) \left\{ f\left[m_1(s)(h_1(s) - 1), s\right] + v(m_1(s)) + \beta g\left[m_2(s)(h_2(s) - 1)\right] + v(m_2(s)) \right\}$$

$$m_0 \left\{ g\left[m_0(h_0 - 1)\right] - v'(m_0) \right\} \leq \beta \sum_s p(s) m_1(s) h_1(s)$$

$$m_1 \left\{ f\left[m_1(s)(h_1(s) - 1), s\right] - v'(m_1(s)) \right\} \leq \beta m_2(s) h_2(s)$$

I proceed to solve for the Ramsey plan assuming an interior solution in the first two periods. From the FOC with respect to h_0, either $m_0 = 0$ or $g'(x_0) = 0$. Under a reasonable parametrization, it will not be the case that the nonmonetary equilibrium is the Ramsey solution.[16] Hence, it has to be the case that $g'(x_0) = 0$, which in turn implies that $x_0 = 0$, and consequently $h_0 = 1$. Intuitively, the Ramsey planner will choose the (inverse of) initial money supply growth to minimize the initial tax distortion, given that h_0 influences the households' payoff but not its intertemporal Euler conditions with consumption risk-neutral preferences.

By differentiating with respect to $h_1(s)$, I find that $f'(x_1(s), s) = -\lambda_0$ where λ_0 is the lagrange multiplier associated with the Euler equation for period $t = 0$.[17] Two main results follow from this equilibrium condition. First, since this equality holds for $s = 1, 2$, it follows that $f'(x_1(1), 1) = f'(x_1(2), 2)$. Essentially, the government wants to smooth tax distortions across states in period 1. Given the functional form for f, it implies that $x_1(2) = (\phi_1/\phi_2)x_1(1)$. Because $\phi_1 < \phi_2$ and tax costs are quadratic in revenues x, we have that $x_1(2) < x_1(1)$ and $c_1(1) < c_1(2)$. The second implication is that $\lambda_0 > 0$ implies that $x_1(s)$ are positive, which is consistent with the idea that the government would want to optimally retire money from circulation, as manifested in $h_1(s) > 1$.

---

[16]Note that if $m_0 = 0$, then for the nonmonetary equilibrium it would also have to be the case that $m_1(s) = m_2(s) = 0$ which is implied by the Euler equation. The nonmonetary competitive equilibrium typically delivers one of the worst utility values for government and households.

[17]Taking FOC with respect to m_0, I obtain the following expression for the lagrange multiplier λ_0

$$\lambda_0 = \frac{v'(m_0)}{1 - v'(m_0) - v''(m_0)m_0}.$$

Using the fact that $f'(x, s)$ is the same across states, it is clear from the FOC with respect to $m_1(s)$ that the government would optimally set the same real money balance in period 1 for both states, i.e. $m_1(1) = m_1(2)$. Hence, given that $x_1(s) = m_1(s)(h_1(s) - 1) > 0$, it follows that $1 < h_1(2) < h_1(1)$. A similar argument can be made to establish that also period-2 allocations $(m_2(s), h_2(s), x_2(s))$ are the same for the two states.

Because in period 1 $c_1(1) < c_1(2)$ while real money balances and continuation values are equal across states, I conclude that the government's utility value is higher in $s = 2$ than in $s = 1$. This latter endogenously determined "bad" state will be of particular interest for the evil agent.

To understand how the presence of uncertainty aversion affects optimal policy, suppose now for a moment that the same allocations as described above were chosen by households when they distrust the approximating model. Given how beliefs distortions, and, hence, the distorted state probabilities are determined in equilibrium (eq. 4), we note that the probabilities will be tilted towards the high-h state in period 1. Consequently, the marginal benefit in period 0 of carrying into the following period an additional unit of money balances exceeds its marginal cost (the right-hand side term of the Euler equation is higher than the left-hand side). The uncertainty averse households respond by increasing their demand for current money balances raising m_0 relative to expected $m_1(s)$. These precautionary savings motives result in higher Ramsey value in an economy with uncertainty averse households (as shown in the general case above) because increased savings (in the form of real money holdings) yield direct utility. Given that in both economies with and without the common beliefs assumption the satiation level for real money holdings is reached in the last period, the government responds to the higher demand for real money holdings by uncertainty averse households with a more gradual deflation process (path for real money holdings) in the economy populated by households faced with model uncertainty. The presence of households' robustness concerns – by resulting in a more gradual path for deflation under Ramsey policy – diminishes government's incentive to deviate when choosing sequentially, thereby effectively alleviating time-consistency problem. It is in that sense that robustness can substitute for (the lack of) government's commitment.[18]

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[18]Of course, for that to be true in a more general case it matters how the government value after deviating changes with the presence of model uncertainty.

In a quantitative example with $\beta = 0.7$, $\psi = 0.01$, $\hat{m} = 100$, $\alpha = 10$, $p = 0.5$, $\phi_2 = 3\phi_1 = 0.045$, I find that under optimal policy real balances increase from 78.4 in period 0 to 84.9 in period 1 with rational-expectation households ($\theta = +\infty$) while with uncertainty-averse households ($\theta = 0.5$) they rise from 81.3 to 84.9 in expectation.[19] The price level drops 11 percent in the former economy and 9 percent in the latter, respectively, reflecting a more gradual deflationary process with uncertainty averse households.

7 Extension: Two-Sided Aversion to Model Uncertainty

So far I have considered a framework where the planner has a unique reference model while the households are faced with model uncertainty and are uncertainty averse. In what follows, I present an economy in which both the public and the government face model uncertainty, potentially to a different extent. I endow the government with multiplier preferences. Let θ^G be the preference-for-robustness parameter describing the extent to which the planner is averse to model uncertainty. Then, the Ramsey problem of uncertainty-averse central banker can be written as

$$
\begin{aligned}
V_t^{G,\theta^G} = \max_{(\mathbf{m},\mathbf{x},\mathbf{h},\mathbf{d},\mathbf{V^H})} \min_{(\mathbf{D^G},\mathbf{d^G})} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) D_t^G(s^t) \{ &\left[u\left(c_t\left(s^t\right)\right) + v(m_t\left(s^t\right)) \right] \\
&+ \theta^G \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}^G(s_{t+1}|s^t) \log d_{t+1}^G(s_{t+1}|s^t) \}
\end{aligned}
$$

$$
\begin{aligned}
\left(\mathbf{m},\mathbf{x},\mathbf{h},\mathbf{d},\mathbf{V^H}\right) &\in CE_{s_0} \\
D_{t+1}^G(s^{t+1}) &= d_{t+1}^G(s_{t+1}|s^t) D_t^G(s^t) \\
\sum_{s_{t+1}} \pi_{t+1}(s_{t+1}|s^t) d_{t+1}^G(s_{t+1}|s^t) &= 1
\end{aligned}
$$

Solving the minimization problem first, one arrives at the following reformulation

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[19]In contrast with the rational expectation economy, real money balances and tax distortions $f'(x, s)$ are not equated any longer across states under the Ramsey plan with uncertainty-averse households. Real money balances in period 1 are 85.40 and 84.35 in states 1 and 2, respectively.

of the problem

$$V_t^{G,\theta^G} = \max_{(\mathbf{m},\mathbf{x},\mathbf{h},\mathbf{d},\mathbf{V^H})} \left\{ u\left(c_t\left(s^t\right)\right) + v\left(m_t\left(s^t\right)\right) - \beta\theta^G \log \sum_{s_{t+1}} \pi\left(s_{t+1}|s_t\right) \exp\left(\frac{-V_{t+1}^{G,\theta^G}\left(s^{t+1}\right)}{\theta}\right) \right\}$$

$$\left(\mathbf{m},\mathbf{x},\mathbf{h},\mathbf{d},\mathbf{V^H}\right) \in CE_{s_0}$$

Now, the continuation value of the government is affected by her own aversion to model uncertainty which has direct consequences for the discussion of time-consistency of government policies. In this economy, the sustainability of equilibrium outcomes is affected by the presence of model uncertainty through two channels. The first one comes from changes in allocations and prices in competitive equilibrium due to uncertainty aversion on the side of the households, as discussed above. The second new channel comes from the incentive compatibility constraint of the government which is now affected by his own uncertainty aversion through continuation values. Fragile beliefs on the side of the planner may help mitigate the time-consistency problem further: by lowering relatively more the value associated with the worst punishment to the planner, planner's fears about model misspecification can help sustain Ramsey outcome in the economy with two-sided aversion to model uncertainty.

8   Conclusion

In this paper I examine how the optimal monetary policies should be designed when the policymaker faces households who cannot form a unique probability model for the underlying state of the economy.

Expectations of future monetary policies influence households' choices of real balances in the current period by affecting the expected value of money in the coming periods. When households exhibit concerns about model misspecification, the effect of the government's policies on the expected value of money is two-fold. Besides their impact on the value of money for every possible future state of the economy, future policies directly influence the households' beliefs about the evolution of exogenous variables, as households base their decisions on the evaluations of worst-case scenarios. It then becomes key for the government to factor in the management of households' expectations when designing monetary policies. Indeed, by internalizing the fact that

households are faced with model uncertainty, the government is able to mitigate the time-consistency problem and to sustain higher values of equilibrium welfare. In other words, robustness to model uncertainty substitutes for commitment.

I provide techniques to fully characterize the sets of all equilibrium outcomes, both with and without commitment on the government's side.[20] To compute these sets, I implement a computational algorithm based on outer hyperplane approximation techniques proposed by Judd, Yeltekin, and Conklin (2003). I am able to solve analytically a simplified three-period version of the model which teaches us about the mechanisms behind the main result: The fact that households are averse towards model uncertainty – by exacerbating their precautionary savings motives and demand for real money balances resulting in a more gradual path for deflation under Ramsey policy – diminishes government's incentive to deviate when choosing sequentially, thereby effectively alleviating time-consistency problem.

While it is true that the way in which attitudes towards model uncertainty is introduced here, following Hansen and Sargent (2008), brings about a form of pessimistic expectations (with respect to the reference model), it is essential to remember that the extent of that pessimism is endogenous to the design of the whole economy.[21] As such, I believe that model uncertainty may play an important role in more general frameworks. As an example, consider an economy modeled within New Keynesian framework and subject to a zero-lower bound. In this framework where a certain positive degree of inflation, rather than deflation, is optimal, worst-case scenarios would be associated with inflation being too low. When the economy is trapped at the zero-lower bound, as discussed in Werning (2011), to stimulate the economy the central bank needs to promise to keep interest rates low for a prolonged period of time. The problem with such a prescription is –again – that of time-inconsistency: once private inflation expectations have risen, the government has an incentive to renege on its original promise. But if agents are uncertainty averse in the way described in this paper, pessimistic ex-

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[20]In an alternative procedure of computing equilibrium outcomes, Feng (2015) delivers equilibrium allocations and prices but relies on assumptions over the equilibrium value correspondence which he is able to verify only numerically.

[21]Naturally, one may be interested to see whether such discrepancies between the planner's and private agents' beliefs can be rationalized empirically as well. To that end, a richer class of monetary models, rather than a stylized model considered here, is needed, and I am currently pursuing this line of research.

pectations would actually mean (endogenously) expecting inflation not rising. As such, the presence of uncertainty averse households may again help the central bank alleviate its time-consistency problem and stimulate the economy.

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A Appendix

A.1 Characterization of the competitive equilibrium sequence

A.1.1 Solving a representative household's maximization problem

Given prices $\{q_t(s^t)\}$, government policies $\{h_t(s^t), x_t(s^t)\}$ and belief distortions $\{D_{t+1}(s^{t+1}), d_{t+1}(s^{t+1})\}_{t=0}^{\infty}$, the households' optimization problem consists of choosing $\{c_t(s^t), M_t(s^t)\}_{t=0}^{\infty}$ and $\{\lambda_t(s^t), \mu_t(s^t)\}_{t=0}^{\infty}$ to maximize and minimize, respectively, the Lagrangian

$$
\begin{aligned}
\mathcal{L}^H &= \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) D_t(s^t) \left\{ \left[u\left(c_t\left(s^t\right)\right) + v(q_t\left(s^t\right) M_t\left(s^t\right)) \right] + \right. \\
&\quad -\lambda_t\left(s^t\right) \left[q_t\left(s^t\right) M_t\left(s^t\right) - y_t\left(s^t\right) + x_t\left(s^t\right) + c_t\left(s^t\right) - q_t\left(s^t\right) M_{t-1}\left(s^{t-1}\right) \right] + \\
&\quad \left. - \mu_t\left(s^t\right) \left[q_t\left(s^t\right) M_t\left(s^t\right) - \overline{m} \right] \right\}.
\end{aligned}
$$

Taking FOCs we obtain

$$
u'(c_t\left(s^t\right)) = \lambda_t\left(s^t\right) \tag{22}
$$

$$
D_t(s^t) \left[v'(m_t\left(s^t\right)) q_t\left(s^t\right) - \lambda_t\left(s^t\right) q_t\left(s^t\right) \right] +
$$
$$
\beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \lambda_{t+1}\left(s^{t+1}\right) D_{t+1}(s^{t+1}) q_{t+1}\left(s^{t+1}\right) - D_t(s^t) \mu_t\left(s^t\right) q_t\left(s^t\right) = 0. \tag{23}
$$

Substitute equation (22) into (23), use (2) and note that $\frac{q_{t+1}\left(s^{t+1}\right)}{q_t(s^t)} = \frac{m_{t+1}\left(s^{t+1}\right) h_{t+1}\left(s^{t+1}\right)}{m_t(s^t)}$

$$
v'(m_t\left(s^t\right)) - u'(c_t\left(s^t\right)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \frac{D_{t+1}(s^{t+1})}{D_t(s^t)} u'(c_{t+1}\left(s^{t+1}\right)) \frac{q_{t+1}\left(s^{t+1}\right)}{q_t\left(s^t\right)} \geq 0,
$$
$$
= 0 \text{ if } m_t\left(s^t\right) < \overline{m}
$$

$$
m_t\left(s^t\right) \left[u'(c_t\left(s^t\right)) - v'(m_t\left(s^t\right)) \right]
$$
$$
-\beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}\left(s_{t+1}|s^t\right) u'(c_{t+1}\left(s^{t+1}\right)) m_{t+1}\left(s^{t+1}\right) h_{t+1}\left(s^{t+1}\right) \leq 0,
$$
$$
= 0 \text{ if } m_t\left(s^t\right) < \overline{m}.
$$

The above expression is the equilibrium condition, equation (9).

A.1.2 Solving alter ego's minimization problem

Given $c_t(s^t), m_t(s^t)$, the evil alter ego's optimization problem consists of choosing $\{D_t(s^t), d_{t+1}(s_{t+1}|s_t)\}$ and $\{\phi_{t+1}(s^{t+1}), \varphi_t(s^t)\}$ to minimize and maximize, respec-

tively, the Lagrangian

$$
\begin{aligned}
\mathcal{L}^{AE} = & \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) D_t(s^t) \{[u(c_t) + v(m_t)] + \\
& + \beta\theta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s_t) \log d_{t+1}(s_{t+1}|s_t)\} + \\
& - \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \phi_{t+1}\left(s^{t+1}\right) \left[D_{t+1}(s^{t+1}) - d_{t+1}(s_{t+1}|s_t) D_t(s^t)\right] + \\
& - \varphi_t\left(s^t\right) \left[\sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s_t) - 1\right].
\end{aligned}
$$

The FOCs for $d_{t+1}(s_{t+1}|s_t)$ and $D_t(s^t)$ are respectively given by

$$
\beta\theta D_t(s^t) \left[\log d_{t+1}(s_{t+1}|s_t) + 1\right] + \beta\phi_{t+1}\left(s^{t+1}\right) D_t(s^t) = \varphi_t\left(s^t\right) \tag{24}
$$

$$
[u(c_t) + v(m_t)] + \beta\theta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s_t) \log d_{t+1}(s_{t+1}|s_t) +
$$

$$
+ \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \phi_{t+1}\left(s^{t+1}\right) d_{t+1}(s_{t+1}|s_t) = \phi_t\left(s^t\right). \tag{25}
$$

Rearranging (24) leads to

$$
\begin{aligned}
\log d_{t+1}(s_{t+1}|s_t) & = -1 + \frac{\varphi_t\left(s^t\right)}{\beta\theta D_t(s^t)} - \frac{\phi_{t+1}\left(s^{t+1}\right)}{\theta} \\
d_{t+1}(s_{t+1}|s_t) & = \exp\left(-1 + \frac{\varphi_t\left(s^t\right)}{\beta\theta D_t(s^t)}\right) \exp\left(-\frac{\phi_{t+1}\left(s^{t+1}\right)}{\theta}\right). \tag{26}
\end{aligned}
$$

By condition (3) it has to be the case that

$$
\exp\left(-1 + \frac{\varphi_t\left(s^t\right)}{\beta\theta D_t(s^t)}\right) \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \exp\left(-\frac{\phi_{t+1}\left(s^{t+1}\right)}{\theta}\right) = 1
$$

$$
\exp\left(-1 + \frac{\varphi_t\left(s^t\right)}{\beta\theta D_t(s^t)}\right) = \frac{1}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \exp\left(-\frac{\phi_{t+1}(s^{t+1})}{\theta}\right)}. \tag{27}
$$

Substituting equation (27) back into (26) yields

$$
d_{t+1}(s_{t+1}|s_t) = \frac{\exp\left(-\frac{\phi_{t+1}\left(s^{t+1}\right)}{\theta}\right)}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \exp\left(-\frac{\phi_{t+1}(s^{t+1})}{\theta}\right)}. \tag{28}
$$

Now we use (24) and impose a respective transversality condition,

$$
\lim_{t\to\infty} \beta^t \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \phi_{t+1}\left(s^{t+1}\right) d_{t+1}(s_{t+1}|s_t) = 0. \tag{29}
$$

It follows that

$$\phi_t\left(s^t\right) = V_t^H\left(s^t\right). \tag{30}$$

Using the above result in equation (28) delivers the equilibrium condition (10)

$$d_{t+1}(s_{t+1}|s_t) = \frac{\exp\left(-\frac{V_{t+1}^H\left(s^{t+1}\right)}{\theta}\right)}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \exp\left(-\frac{V_{t+1}^H\left(s^{t+1}\right)}{\theta}\right)}.$$

A.1.3 On the transversality condition

We will show that the transversality condition,

$\beta^t \sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) u'\left[(f(x_t\left(s^t\right), s_t)\right] m_t\left(s^t\right) h_t\left(s^t\right) \to 0$ as $t \to \infty$ for all t and all s^t, is satisfied if $(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}) \in E^\infty$.

Since E is compact, for any $\left(x_t\left(s^t\right), m_t\left(s^t\right), h_t\left(s^t\right), d_{t+1}(s_{t+1}|s^t)\right) \in E$, it must be that $\sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) u'\left[(f(x_t\left(s^t\right), s_t)\right] m_t\left(s^t\right) h_t\left(s^t\right)$ belongs to a compact interval (due to continuity of u' and f) for every t. Hence, it has to be that $\sum_{s_{t+1}} \pi(s_{t+1}|s_t) d_{t+1}(s_{t+1}|s^t) u'\left[(f(x_t\left(s^t\right), s_t)\right] m_t\left(s^t\right) h_t\left(s^t\right)$ is a bounded sequence, and the required sequence indeed converges to zero.

A.2 Example of competitive equilibrium sequences

Assume that $s_t = H, L$ and that the production function is such that $f(0, H) = f(0, L)$. Set $(\mathbf{m}, \mathbf{x}, \mathbf{h}) = \{m^*, 0, 1\}_{t=0}^\infty$ where m^* satisfies the following condition for all t and all s_t

$$u'\left(f(0, s_t)\right)(1 - \beta) = v'\left(m^*\right).$$

Then $(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE_s$.

A.3 Proof of Corollary 3.

CE_s for all $s \in \mathbb{S}$ is compact.

Proof. Fix $s_0 \in \mathbb{S}$. Let $\left(\mathbf{m}^n, \mathbf{x}^n, \mathbf{h}^n, \mathbf{d}^n, \mathbf{V^{H^n}}\right)$ be the sequence from $CE_{s=s_0}$ converging to some sequence $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right)$. We need to show that this limiting sequence belongs to $CE_{s=s_0}$.

$CE_{s=s_0}$ is a nonempty subset of a compact set $\mathbb{E}^\infty$. Since $\mathbb{E}^\infty$ is compact, it is closed, and, hence, $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in \mathbb{E}^\infty$.

The fact that $\left(\mathbf{m}^n, \mathbf{x}^n, \mathbf{h}^n, \mathbf{d}^n, \mathbf{V^{H^n}}\right) \in CE_{s=s_0}$ implies that equations (9) - (12) are satisfied for each n. Consequently, by continuity of u, v, u', v' and f, $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right)$ satisfy these same equations. It follows then from Proposition 1 that $\left(\mathbf{m}, \mathbf{x}, \mathbf{h}, \mathbf{d}, \mathbf{V^H}\right) \in CE_{s=s_0}$, which means that $CE_{s=s_0}$ is a closed subset of the compact set. Hence, it is compact. $\qquad\square$

A.4   Solving the time-0 Ramsey Problem

Proof. The FOC for the Ramsey problem at time $t = 0$ boils down to

$$f'(x_0)(u'(c_0) - \phi_0 u''(c_0)m_0) = 0 \tag{31}$$

where ϕ_0 is a Lagrange multiplier on condition (20). Hence, it must be either that $f'(x_0) = 0$ or that $u'(c_0) = 0$ and $m_0 = 0$ hold at the same time (since $\phi_0 > 0$ and $u''(c_0) < 0$). We will show that the latter condition cannot be simultaneously satisfied. Notice that $m_0 = 0$ implies that $x_0 = 0$. But then $u'(f(x_0 = 0))$ cannot be equal to zero.

Next, rewrite condition (20) using the condition (17) as

$$v'(m_0) = u'(f(0, s)) - \beta \sum_{s'} \pi(s'|s) \frac{\exp\left(-\frac{V^{H\prime}_{s'}}{\theta}\right)}{\sum_{s'} \pi(s'|s)\exp\left(-\frac{V^{H\prime}_{s'}}{\theta}\right)} u'(f(x', s'))h', \quad \leq \text{ if } m = \overline{m}. \tag{32}$$

where the prime notation refers to $t = 1$ values.

All else equal, the second - expectation - term on the RHS is lower in the economy with rational expectations. (Recall that the presence of fragile beliefs manifests itself in a pessimistic exponential twisting of the likelihood). That is, it must be that $v'(m_0) > v'(m_0^{RE})$, and, hence, $m_0 > m_0^{RE}$. $\qquad\square$

A.5   Numerical Algorithm: Outer Hyperplane Approximation

Table 1: Monotone Outer Hyperplane Approximation

Step 0:	Approximate each $Z_0(s) \supset \Lambda(s)$.

Step 0: Approximate each $Z_0(s) \supset \Lambda(s)$.

For each $s = 1, ..., S$, and $g_l \in G$, $l = 1, ..., D$, compute

$$c_{l,s}^0 = \max g_{l,1} V_s^G + g_{l,2} V_s^H + g_{l,3} \mu_s, \qquad \text{such that}$$

$$(V_s^G, V_s^H, \mu_s) \in Z_0(s).$$

Let $C_s^0 = \{c_{1,s}^0, ..., c_{D,s}^0\}$ for $s = 1, ..., S$.

Step 1: Given C_s^k for $s = 1, ..., S$, update C_s^{k+1}.

For each $s = 1, ..., S$, and $g_l \in G$, $l = 1, ..., D$,

(a) For each pair (m, h), solve

$$P_s^k(m, h) = \min_{(V^{G\prime}, V^{H\prime}, \mu\prime)} u[f(x, s)] + v(m) + \beta \sum_{s'\in\mathbb{S}} \pi(s'|s) V_{s'}^{G\prime},$$

such that $m[u'(f(x, s)) - v'(m)] = \beta \sum_{s'\in\mathbb{S}} \pi(s'|s) d_{s'}' \mu_{s'}'$ with $\leq$ if $m = \overline{m}$

$$x = m(h - 1)$$

$$g_l \cdot (V_{s'}^{G\prime}, V_{s'}^{H\prime}, \mu_{s'}') \leq c_{l,s'}^k \qquad \text{for } s' = 1, ..., S, \, l = 1, ..., D.$$

Let $P_s^k(m, h) = +\infty$ if no $(V^{G\prime}, V^{H\prime}, \mu')$ satisfies the constraints.

Let $R_s^k(h) = \min_m P_s^k(m, h)$. Let $\underline{V}_s^G = \max_{h\in\Pi} R_s^k(h)$.

(b) For each pair (m, h), solve

$$c_{l,s}^{k+1}(m, h) = \max_{(V^{G\prime}, V^{H\prime}, \mu')} g_{l,1} V_s^G + g_{l,2} V_s^H + g_{l,3} \mu_s, \qquad \text{(P1)}$$

such that $V_s^G = u[f(x, s)] + v(m) + \beta \sum_{s'\in\mathbb{S}} \pi(s'|s) V_{s'}^{G\prime}$

$$V_s^H = u[f(x, s)] + v(m) - \beta\theta \log \sum_{s'\in\mathbb{S}} \pi(s'|s) \exp\left\{-V_{s'}^{H\prime}/\theta\right\}$$

$$\mu_s = u'[f(x, s)](m + x)$$

$$m[u'(f(x, s)) - v'(m)] = \beta \sum_{s'\in\mathbb{S}} \pi(s'|s) d_{s'}' \mu_{s'}' \quad \text{with} \leq \text{if } m = \overline{m}$$

$$x = m(h - 1)$$

$$d_{s'}' = \exp\left\{-V_{s'}^{H\prime}/\theta\right\} / \sum_{s'\in\mathbb{S}} \pi(s'|s) \exp\left\{-V_{s'}^{H\prime}/\theta\right\}$$

$$V_s^G \geq \underline{V}_s^G$$

$$g_l \cdot (V_{s'}^{G\prime}, V_{s'}^{H\prime}, \mu_{s'}') \leq c_{l,s'}^k \qquad \text{for } s' = 1, ..., S, \, l = 1, ..., D.$$

where $c_{l,s}^{k+1}(m, h) = -\infty$ if no $(V^{G\prime}, V^{H\prime}, \mu')$ satisfies the constraints.

Let $(V^{G\prime}, V^{H\prime}, \mu')_{l,s}(m, h) \in \mathbb{R}^{S \times 3}$ be the solution to (P1).

(c) For each $s = 1, ...S$, and $l = 1, ..., D$, define

$$c_{l,s}^{k+1} = \max_{(m,h)} c_{l,s}^{k+1}(m, h)$$

$$(m^*, h^*)_{l,s} = \arg\max_{(m,h)} c_{l,s}^{k+1}(m, h)$$

Update C_s^{k+1} as $C_s^{k+1} = \{c_{1,s}^{k+1}, ..., c_{D,s}^{k+1}\}$ for $s = 1, ..., S$

Step 2: Stop if $\max_{l,s} |c_{l,s}^{k+1} - c_{l,s}^k| < 10^{-6}$; otherwise go to Step 1.